On the Fields of Some Brownian Martingales

by

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1. Introduction

Suppose $(\Omega, \mathcal{F}, P)$ is a probability space, and $\{B_t\}_{t \geq 0}$ is a standard 1-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_t^B\}_{t \geq 0}$ be the right continuous, P-complete σ-fields generated by $\{B_t\}_{t \geq 0}$. If $\{M_t, \mathcal{F}_t^B\}_{t \geq 0}$ is a martingale with $EM_t^2 < \infty$ for all t, it is well-known that there exists an $\{\mathcal{F}_t^B\}_{t \geq 0}$-adapted process $\{C_t\}_{t \geq 0}$ with $E\int_0^t C_s^2\, ds < \infty$ for all t, such that $\{\int_0^t C_s\, dB_s\}_{t \geq 0}$ is a version of $\{M_t\}_{t \geq 0}$ (see, for example, Kallianpur (1977)). In particular, every such M has a version with continuous paths, a fact which implies that every $\{\mathcal{F}_t^B\}_{t \geq 0}$-stopping time is predictable, a strong regularity property of the process B. (see Chung and Walsh (1974)).

Now let $\{\mathcal{F}_t^M\}_{t \geq 0}$ be the right continuous, P-complete σ-fields generated by $\{M_t\}_{t \geq 0}$. Clearly $\mathcal{F}_t^M \subset \mathcal{F}_t^B$ for all t: call M and B <u>equivalent</u> if $\mathcal{F}_t^M = \mathcal{F}_t^B$ for all t. The questions we consider in this paper (and answer, for some special classes of integrands described later) are:

1) What conditions on $\{C_t\}_{t \geq 0}$ guarantee that M be equivalent to B?

2) More generally, when can we find some $\{\mathcal{F}_t^B\}_{t \geq 0}$-adapted Brownian Motion $\{X_t\}_{t \geq 0}$ such that M is equivalent to X?

3) If no such X exists, how "bad" are the fields $\{\mathcal{F}_t^M\}_{t \geq 0}$ -- in particular, do they support discontinuous martingales?

In the next section, we show that these questions are easily settled for two important classes of integrands, simple functionals and nonrandom functions. We also present some examples indicating the complications which can arise with more complicated integrands. The remainder of the paper is devoted to answering the questions for integrands of the form $C_s = f(B_s)$ where f is continuous with non-accumulating zero set.

2. Some examples

First, suppose there is a finite time set $\{0 = t_1 < \ldots < t_n < \infty\}$ such that $C_t(\omega) = \sum_{i=1}^{n} C_{t_i}(\omega)\, 1_{[t_i,\, t_{i+1})}(t)$ where C_{t_i} is $\mathfrak{F}^B_{t_i}$-adapted and $EC^2_{t_i} < \infty$, $1 \leq i \leq n$. Then, if $P(C_{t_i} = 0) = 0$, $1 \leq i \leq n$, M is equivalent to B; if for some i, $P(C_{t_i} = 0) > 0$, M is equivalent to no Brownian motion. All stochastic integrals are obtained as limits of integrals with such integrands (which may be taken to satisfy $P(C_{t_i} = 0) = 0$, $1 \leq i \leq n$): hence every L^2-martingale on $\{\mathfrak{F}^B_t\}_{t \geq 0}$ may be approximated arbitrarily closely (uniformly on almost all paths) by martingales equivalent to B.

Next, consider the case of nonrandom integrands: $C_t(\omega) = f(t)$, with $\int_0^t f^2(s)ds < \infty$ for all t. The Gaussian martingale $\{M_t = \int_0^t f(s)\, dB(s)\}_{t \geq 0}$ is then equivalent to B if and only if $\text{leb}\{s:f(s) = 0\} = 0$ ("leb" is Lebesgue measure). If so, $B_t = \int_0^t \frac{1}{f(s)}\, dM(s)$ and so $\mathfrak{F}^B_t \subset \mathfrak{F}^M_t$, $t \geq 0$. If not (and $\text{leb}\{s \leq t:f(s) = 0\} < t$) the nontrivial Gaussian processes $\{M_t\}_{t \geq 0}$ and $\{Y_t = \int_0^t 1_{\{f(s) = 0\}}\, dB(s)\}_{t \geq 0}$ are independent, so Y_t is <u>not</u> $\mathfrak{F}^M_t$-measurable and $\mathfrak{F}^M_t \underset{\neq}{\subset} \mathfrak{F}^B_t$ as soon as $\text{leb}\{s \leq t:f(s) = 0\} > 0$.

If $\text{leb}\{s:f(s) = 0\} > 0$, the fields $\{\mathfrak{F}^M_t\}_{t \geq 0}$ are not generated by <u>any</u> Brownian motion, but they are still quite well-behaved, supporting no discontinuous martingales. Both facts follow from the integral representation theorem given below; the proof of which exactly follows a proof of Kallianpur (1977) for the Brownian case $(f(t) \equiv 1)$. (For suppose M <u>were</u> equivalent to a Brownian Motion X. Then $X_t = \int_0^t D(s)\, dM(s) = \int_0^t D(s)f(s)\, dB(s)$ and so $t = \langle X \rangle_t = \int_0^t D^2(s)f(s)\, ds$. But then for almost every path, $D^2(s)f^2(s) = 1$ a.s. (leb), a contradiction since $\text{leb}\{s:f(s) = 0\} > 0$.)

<u>Theorem</u>: If $M_t = \int_0^t f(s)\, dB(s)$ where f is a nonrandom locally L^2-function, then every L^2-martingale with respect to $(\{\mathcal{F}_t^M\}_{t \geq 0}, P)$ can be represented as a stochastic integral of M -- that is, if Y is such a martingale, then there exists an $\{\mathcal{F}_t^M\}_{t \geq 0}$-adapted process $\{C_t\}_{t \geq 0}$ with $E\int_0^t C_s^2 ds < \infty$ for all t, such that $\{\int_0^t C_s\, dM(s)\}_{t \geq 0}$ is a version of Y.

When the integrands are random and not simple, the situation becomes much more complicated as the following examples show:

1) Let $\tau = \inf\{s: |B_s| = 1\}$. Set $C_s = 1_{\{\tau < s\}}$. Then the random variable $X(\omega) = \mathrm{leb}\{s: C_s(\omega) = 0\}$ is positive a.s. As was true with nonrandom and simple integrands above, this implies that $M = \int C\, dB$ is not equivalent to any Brownian Motion. But here there is no integral representation theorem. Since $\tau = \inf\{s: M_s \neq 0\}$, τ is an $\{\mathcal{F}_t^M\}_{t \geq 0}$-stopping time. Moreover, τ is clearly $\{\mathcal{F}_t^M\}_{t \geq 0}$-unpredictable, and the martingale $\{E(\tau | \mathcal{F}_t^M)\}_{t \geq 0}$ is discontinuous.

2) Consider the integrand B_s, so $P\{\mathrm{leb}\{s: B_s = 0\} = 0\} = 1$. However, $M_t = \int_0^t B_s\, dB_s = \dfrac{B_t^2 - t}{2}$, and so M is <u>not</u> equivalent to B (but to $|B|$). As will be shown below, there <u>is</u> a Brownian Motion X which is equivalent to M. But as we shall also see, there exist integrands $\{D_t\}_{t \geq 0}$ such that $P\{\mathrm{leb}\{s: D_s = 0\} = 0\} = 1$ and $\{\int_0^t D_s\, dB_s\}_{t \geq 0}$ is equivalent to <u>no</u> Brownian motion and generates fields which support discontinuous martingales.

3. Some definitions and the statement of the theorem.

Let f be a continuous function. Define the zero set, Z_f, of f and the crossing set, C_f, of f as follows:

$$Z_f = \{x : f(x) = 0\}$$

$$C_f = \{x : f(x) = 0 \text{ and } \lim_{s \downarrow x} \operatorname{sgn} f(s) \neq \lim_{s \uparrow x} \operatorname{sgn} f(s)\},$$

where $\operatorname{sgn} f = 1_{\{f > 0\}} - 1_{\{f < 0\}}$.

For x in C_f, define

$$\gamma_f(x) = \inf\{s \geq 0 : f(x + s) \neq -f(x-s)\}.$$

Let $\gamma C_f = \{s : s = \gamma_f(x) \text{ for some } x \text{ in } C_f\}$.

Theorem: Suppose f is a continuous function with $Z_f = C_f$ and $\operatorname{Card}(C_f) < \infty$. Suppose also that for all finite t, $E\int_0^t f^2(B_s)\,ds < \infty$. Let $M_t^f = \int_0^t f(B_s)\,dB_s$. Then:

1) If $\gamma C_f = \{0\}$, M^f is equivalent to B.

2) If $\gamma C_f = \{0\} \cup \{\infty\}$, M^f is equivalent to a Brownian Motion X, which is itself equivalent to B reflected in a (perhaps infinite) interval.

3) If $\gamma C_f \cap (0, \infty) \neq \emptyset$, then M^f is equivalent to no Brownian Motion, and $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$ supports discontinuous martingales.

In section 4, three basic lemmas are established; the theorem itself is proved in sections 5-7.

4. Three lemmas

The first lemma shows that, if for each path the integrand vanishes only on a set of zero Lebesgue measure, the fields of the integral are sufficiently rich to support a Brownian Motion.

Lemma 1: Suppose $\{C_t\}_{t \geq 0}$ is $\{\mathcal{F}_t^B\}_{t \geq 0}$-adapted, $E\int_0^t C_s^2\, ds < \infty$ for all t, and $P\{\text{leb}\{s: C_s = 0\} = 0\} = 1$. Then there exists a Brownian Motion $\{X_t\}_{t \geq 0}$ such that for all t, $\mathcal{F}_t^X \subseteq \mathcal{F}_t^{\int C\, dB} \subseteq \mathcal{F}_t^B$.

Proof: By Ito's formula, (see, for example, McKean (1969)), $\int M\, dM = \frac{M^2 - \langle M \rangle}{2}$ where $\langle M \rangle_t = \int_0^t C_s^2\, ds$, so $\langle M \rangle = M^2 - 2\int M\, dM$ is $\{\mathcal{F}_t^M\}_{t \geq 0}$-adapted. Let $D_t(\omega) = \overline{\lim_{h \downarrow 0}} \frac{\langle M \rangle_{t+h} - \langle M \rangle_t}{h}(\omega)$ -- then D is $\{\mathcal{F}_t^M\}_{t \geq 0}$-adapted. Moreover, for almost all ω, $D_t(\omega) = C_t^2(\omega)$ a.s. (Lebesgue). For almost all ω, then, $D_t(\omega) \neq 0$ a.s. (Lebesgue).

Let $F_t(\omega) = |D_t|^{-\frac{1}{2}}(\omega)$: so for almost all ω, $F_t(\omega) = |C_t|^{-1}$ a.s. (Lebesgue). Since for $s < \infty$, $E\int_0^s \frac{C_t^2}{|C_t|^2}(\omega)\, dt = s < \infty$, $E\int_0^s F_t^2(\omega) C_t^2(\omega)\, dt < \infty$ also, and so $\{X_t = \int_0^t F(s)\, dM(s) = \int_0^t F(s) C(s)\, dB(s)\}_{t \geq 0}$ exists and is, of course, $\{\mathcal{F}_t^M\}_{t \geq 0}$-adapted. Since $E\int_0^t |F_s C_s(\omega) - \frac{C_s}{|C_s|}(\omega)|\, ds = 0$, for each t $X_t = \int_0^t \frac{C_s}{|C_s|}\, dB(s)$ a.s. (P) and thus for all t, $\langle X \rangle_t = \int_0^t \frac{C_s^2}{C_s^2}\, ds = t$ a.s. (P). Hence X is a Brownian Motion (see McKean (1969)).

The next lemma is fundamental. Let $M(t) = \int_0^t \operatorname{sgn} B(s)\, dB(s)$. An easy application of Ito's formula shows that $P(M(t) = |B(t)| - \ell_0(t)\ \forall t) = 1$ where $\ell_0(t, \omega) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \text{leb}\{s \leq t: |B_s(\omega)| \leq \varepsilon\}$ is the local time of B at 0. Since $\{\ell_0(t)\}_{t \geq 0}$ is adapted to $\{\mathcal{F}_t^{|B|}\}_{t \geq 0}$, M is $\{\mathcal{F}_t^{|B|}\}_{t \geq 0}$-adapted also. The converse is also true.

<u>Lemma 2</u>: Let $M_t = \int_0^t \operatorname{sgn} B(s)\, dB(s)$. Then M is equivalent to $|B|$.

<u>Proof</u>: Set $A = \{\omega: \forall t,\ \ell_0(t) < \infty$ and $M(t) = |B(t) - \ell_0(t)|\}$; then $P(A) = 1$. For ω in A, $\ell_0(\omega, \cdot)$ is an increasing function, and the measure $d\,\ell_0(\omega, \cdot)$ is concentrated on $Z_{B\cdot(\omega)}$.

Fix t. Let $s' = \max\{s \leq t{:}B_s(\omega) = 0\}$. Then $r > s'$ implies that $\ell_0(r, \omega) = \ell_0(s', \omega)$; and since $|B_{s'}(\omega)| = 0$, $M_{s'}(\omega) < M_r(\omega)$. If $r < s$, then $\ell_0(r, \omega) \leq \ell_0(s', \omega)$, and so $M_r(\omega) \geq M_{s'}(\omega)$. Hence $s' = \max\{s \leq t{:}M_s(\omega) = \min_{0\leq r\leq t} M_r(\omega)\}$. Since $|B_t|(\omega) = M_t(u) - M_{s'}(\omega)$, we have recovered $|B_t|(\omega)$ from $\{M_s(\omega)\}_{0\leq s\leq t}$. Since $\mathcal{F}_t^M$ is complete, $|B_t|$ is $\mathcal{F}_t^M$-measurable.

Suppose $\{X_t\}_{t \geq 0}$ is a stochastic process on $(\Omega, \{\mathcal{G}_t\}_{t \geq 0}, P)$, and $\tau_1 \leq \tau_2$ are $\{\mathcal{G}_t\}_{t \geq 0}$-stopping times. For each t, let $Y_t = X_{t\wedge\tau_2} - X_{t\wedge\tau_1}$, so that $\{Y_t\}_{t \geq 0}$ is also a stochastic process on $(\Omega, \{\mathcal{G}_t\}_{t \geq 0}, P)$. Set $\mathcal{F}^X_{(\tau_1,\tau_2)} = \bigvee_{t\geq 0} \mathcal{F}^Y_t$.

<u>Lemma 3</u>: Suppose X and M are processes on $(\Omega, \{\mathcal{G}_t\}_{t \geq 0}, P)$, $\{\mathcal{G}_t\}_{t \geq 0}$ are right continuous, $\mathcal{G}_0$ is trivial and X has continuous paths. Suppose that $\{\tau_n\}_{n \geq 0}$ is a sequence of $\{\mathcal{F}_t^M\}_{t > 0}$-stopping times, $\tau_0 = 0$ and $\tau_n \uparrow \infty$. If for all $n \geq 0$ and $s \leq t$, $\mathcal{F}^X_{(\tau_n\wedge s,\ \tau_{n+1}\wedge s)} \subset \mathcal{F}^M_{(\tau_n\wedge s,\ \tau_{n+1}\wedge s)}$, then $\mathcal{F}_t^X \subset \mathcal{F}_t^M$.

<u>Proof</u>: Since X has continuous paths and $\tau_n \to t$, for $s \leq t$, $X_{\tau_n\wedge s} \to X_s$; hence it suffices to show that $X_{\tau_n\wedge s}$ is $\mathcal{F}_t^M$-measurable.

$X_{\tau_n\wedge s} = X_{\tau_0} + (X_{\tau_1\wedge s} - X_{\tau_0}) + \ldots + (X_{\tau_n\wedge s} - X_{\tau_{n-1}\wedge s})$, and by definition, $(X_{\tau_{k+1}\wedge s} - X_{\tau_k\wedge s})$ is $\mathcal{F}^X_{(\tau_k\wedge s,\ \tau_{k+1}\wedge s)}$-measurable -- and th s, by assumption, is $\mathcal{F}^M_{(\tau_k\wedge s,\ \tau_{k+1}\wedge s)}$-measurable also. Since each τ_k is an $\{\mathcal{F}_t^M\}_{t \geq 0}$-stopping time,

$\tau_k \wedge s \leq s$, $\mathcal{F}^M_{(\tau_k \wedge s,\ \tau_{k+1} \wedge s)} \subset \mathcal{F}^M_t$. Thus $X_{\tau_n \wedge s}$ is $\mathcal{F}^M_t$-measurable, and so $\mathcal{F}^X_t \subset \mathcal{F}^M_t$.

5. Proof of theorem: $\gamma C_f = \{0\}$.

For simplicity, we divide the argument into two cases.

Case 1: $\gamma^{-1}\{0\} = 0$.

Set $X_t = \int_0^t \operatorname{sgn} f(B_s)\, dB_s$. By Lemma 1, X_t is $\mathcal{F}_t^{M^f}$-measurable. Since $\operatorname{sgn} f(B_s) = (\lim_{x\downarrow 0} \operatorname{sgn} f(x)) \operatorname{sgn} B(s)$, X is equivalent to $\int \operatorname{sgn} B(s)\, dB(s)$, and hence, by Lemma 2, to $|B|$. Thus $|B|_t$ is $\mathcal{F}_t^{M^f}$-measurable for each t.

Since $\langle M^f\rangle_t = \int_0^t f^2(B_s)\, ds$ and f is continuous, $\langle M^f\rangle_t' = f^2(B_t)$ for all t and so $|f(B_t)|$ is $\mathcal{F}_t^{M^f}$-measurable for all t.

Fix t. Let $s'(\omega) = \max\{s \leq t : |B(s)| = 0\}$. Then s' is $\mathcal{F}_t^{M^f}$-measurable. For s in $(s'(\omega), t)$, either A): $B(s) = +|B(s)|$

or B): $B(s) = -|B(s)|$.

Since $\gamma(0) = 0$ and $|B(s'(\omega))| = 0$, for almost all ω we can find $\tilde{s}(\omega)$ in $(s'(\omega), t)$ such that

$$f(|B_{\tilde{s}}(\omega)|) \neq -f(-|B_{\tilde{s}}(\omega)|) \text{ -- that is,}$$

$$|f(|B_{\tilde{s}}(\omega)|)| \neq |f(-|B_{\tilde{s}}(\omega)|)|.$$

Comparing $|B_{(\tilde{s})}(\omega)|$ with $|f(B_{(\tilde{s})}(\omega)|$ will therefore allow us to determine which of A or B is true and hence we may determine $B_t(\omega)$. So $\mathcal{F}_t^B \subset \mathcal{F}_t^{M^f}$, and M^f is equivalent to B.

Case 2: General $\gamma^{-1}\{0\}$.

Let $\tau_0 \equiv 0$, $\tau_{n+1} = \inf\{t > \tau_n : B_t \in \{C_f - B(\tau_n)\}\}$ for $n = 0, 1, 2, \ldots$

Claim: For all n and t, $\mathcal{F}^B_{(0,\ \tau_n \wedge t)} \subset \mathcal{F}^{M^f}_{(0,\ \tau_n \wedge t)}$. Once the claim is established, an easy modification of lemma 3 shows that, if τ_n is an $\{\mathcal{F}_t^M\}_{t \geq 0}$-stopping time, $\mathcal{F}_t^B = \mathcal{F}_t^{M^f}$.

<u>Proof of claim:</u>

1) Let $z_1 = \max(z\epsilon C_f \cap (-\infty, 0))$ and $z_2 = \min(z\epsilon C_f \cap (0, \infty))$,

$$g(x) = \begin{cases} f(x) & \text{for } x \text{ in } [z_1, z_2), \\ > 0 \text{ and continuous} & \text{for } x \text{ not in } [z_1, z_2], \end{cases}$$

and $X_t = \int_0^t g(B_s)\, dB_s$. Then $M_{s\Lambda\tau_1} = X_{s\Lambda\tau_1}$ for all s. If $0 \notin C_f$, we may apply Lemma 1 to conclude that X is equivalent to B. If $0 \in C_f$, Case 1 above implies that X is equivalent to B. In either case, $\mathcal{F}^M_{(0,\ \tau_1\Lambda t)} = \mathcal{F}^X_{(0,\ \tau_1\Lambda t)} = \mathcal{F}^B_{(0,\ \tau_1\Lambda t)}$.

2) Assume for all t, $\mathcal{F}^M_{(0,\ \tau_n\Lambda t)} = \mathcal{F}^B_{(0,\ \tau_n\Lambda t)}$. Let $Y_s = \int_0^s \operatorname{sgn} f(B_s)\, dB_s$.

By Lemma 1, for each s Y_s is $\mathcal{F}^{M_f}_s$-measurable, and so the process $\{Z_s = Y_{\tau_{n+1}\Lambda s} - Y_{\tau_n\Lambda s} = \int_{\tau_n\Lambda s}^{\tau_{n+1}\Lambda s} \operatorname{sgn} f(B_r)\, dB_r\}_{0 \le s \le t}$ is $\mathcal{F}^{M_f}_{(0,\ \tau_{n+1}\Lambda t)}$-measurable. Note that $f(B(\tau_n)) = 0$ and for r in (τ_n, τ_{n+1}), $\operatorname{sgn} f(B(r)) =$

$$\begin{cases} C(B_{(\tau_n)}) \text{ if } B(r) > B(\tau_n) \\ -C(B_{(\tau_n)}) \text{ if } B(r) < B(\tau_n) \end{cases}$$

where C is a function from $C_f \to \{-1, +1\}$.

Let $\tilde{B}_s = B_{\tau_n+s} - B_{\tau_n}$. Then $\{\tilde{B}_s\}_{s \ge 0}$ is a Brownian Motion, and by Lemma 2, $\int \operatorname{sgn} \tilde{B}\, d\tilde{B}$ is equivalent to $|B| = |B_{\tau_n+\cdot} - B_{\tau_n}|$. Set

$\tilde{Z}_s = Z_{\tau_n+s} \cdot 1_{(0,\ \tau_{n+1} - \tau_n)}(s)$. Then $\tilde{Z}_s = 1_{(0,\tau_{n+1}-\tau_n)}(s) \int_0^s \operatorname{sgn} f(B_{\tau_n+r})\, dB(\tau_n+r)$

$= 1_{(0,\ \tau_{n+1}-\ \tau_n)}(s) \cdot C(B_{(\tau_n)}) \int_0^s \operatorname{sgn} (\tilde{B}_r)\, d\tilde{B}_r$. Hence $\{|B_{\tau_{n+1}\Lambda s} - B_{\tau_n\Lambda s}|\}$ is

$\mathcal{F}^{M^f}_{(0,\ \tau_{n+1}\Lambda t)}$-measurable.

Since $\gamma(B(\tau_n)) = 0$, and by assumption $B(\tau_n \wedge t)$ is $\mathcal{F}^{M^f}_{(0,\ \tau_n \wedge t)}$-measurable, the argument of Case 1 shows that $\{\mathrm{sgn}(B_{\tau_n \wedge s})\}_{0 \leq s \leq t}$ is $\mathcal{F}^{M^f}_{(0,\ \tau_{n+1} \wedge t)}$-measurable, and we may conclude that $\mathcal{F}^{B}_{(0,\ \tau_{n+1} \wedge t)} \subset \mathcal{F}^{M^f}_{(0,\ \tau_{n+1} \wedge t)}$ for all t, and hence that τ_n is an $\{\mathcal{F}^{M^f}_t\}_{t \geq 0}$-stopping time for each n. Thus $\mathcal{F}^{M^f}_t = \mathcal{F}^{B}_t$ for all t.

6. Proof of theorem: $\gamma C_f = \{\infty\} \cup \{0\}$.

Case 1: $z_0 = \gamma^{-1}\{\infty\}$.

Define the Brownian Motion $\{Y_t = \int_0^t \operatorname{sgn}(B_s - z_0)\, dB_s\}_{t \geq 0}$. Let $\tau = \inf(t:B_t = z_0)$. By an easy modification of Lemma 2, Y is equivalent to X where $X_t = (B_t - z_0)1_{(\tau \geq t)} + |B_t - z_0|1_{(\tau < t)}$.

Claim: M^f is equivalent to Y.

Proof of claim:

Since $\gamma(z_0) = \infty$, for all $x \geq 0$ $f(z_0 + x) = -f(z_0 - x)$. We may choose a sequence of functions $\{f_n\}$ $n = 1, 2, \ldots$ satisfying the following three conditions:

i) f_n is continuously differentiable

ii) $f_n(z_0 + x) = -f_n(z_0 - x)$ for all $x > 0$

iii) for $|z| \leq n$, $|f(z) - f_n(z)| \leq 1/n$ and $|f_n(z)| \leq 2|f(z)|$.

Fix $t < \infty$: then

$$E\int_0^t |f_n(B_s) - f(B_s)|^2\, ds = E\int_0^t 1_{\{B_s \leq n\}} |f_n(B_s) - f(B_s)|^2\, ds$$

$$+ E\int_0^t 1_{\{B_s > n\}} |f_n(B_s) - f(B_s)|^2\, ds$$

$$\leq \frac{t}{n^2} + 2E\int_0^t 1_{\{B_s > n\}} |f(B_s)|^2\, ds.$$

Since $E\int_0^t |f(B_s)|^2\, ds < \infty$, the last term $\to 0$ as $n \to \infty$, and so $\mathcal{F}_t^{M^f} \subset \bigcup_n \mathcal{F}_t^{M^{f_n}}$. So it suffices to show, for h satisfying conditions i-iii, $\mathcal{F}_t^{M^h} \subset \mathcal{F}_t^Y$.

For such an h, let $F(x) = \begin{cases} \int_{z_0}^x h(y)\, dy & x \geq z_0 \\ F(2z_0 - x) & x < z_0. \end{cases}$

Then $F' = h$, and there exist bounded functions G and g defined on R^+ such that $F(x) = G|x-z_0|$ and $F''(x) = g|x-z_0|$. We may apply Ito's formula to obtain:

$$M_t^h = G|B_t - z_0| - G|z_0| - \tfrac{1}{2}\int_0^t g|B_s - z_0|\, ds,$$

so M_t^h is $\mathfrak{F}_t^X$-measurable.

<u>$\mathfrak{F}_t^X \subset \mathfrak{F}_t^{M^f}$</u>

1) Suppose $C_f = \{0\}$ and $\gamma(0) = \infty$. Then $\operatorname{sgn} f(B_s) = (\lim_{x \downarrow 0} \operatorname{sgn} f(x)) \operatorname{sgn} B_s$, so $\int \operatorname{sgn} f(B_s)\, dB_s$ is equivalent to $|B|$ by Lemma 2, and by Lemma 1, $\int \operatorname{sgn} f(B_s)\, dB(s)$ is adapted to $\{\mathfrak{F}_t^{M^f}\}_{t \geq 0}$. Hence $\mathfrak{F}_t^{|B|} \subset \mathfrak{F}_t^{M^f}$ for all $t \geq 0$.

2) Next, suppose $C_f = \{0\} \cup \gamma^{-1}(0)$, and $\gamma(0) = \infty$. Let $x_1 = \min\{x > 0, x \in C_f\}$ and set $\tau_1 = \inf(t: |B_t| = x_1)$. For $k = 1, 2, \ldots$, set $\tau_{2k} = \inf(t > \tau_{2k-1}: B_t = 0)$ and $\tau_{2k+1} = \inf(t > \tau_{2k}: |B_t| = x_1)$. We now apply Lemma 3:

Define bounded continuous g with $g = f$ on $(-x_1, x_1)$, and $g(s) = -g(-s)$ for all s, and $C_g = \{0\}$. Then $\gamma(0) = \infty$, and by 1) above, M^g is equivalent to $|B|$. Since $\mathfrak{F}_{(0,\ \tau_1 \wedge t)}^{M^f} = \mathfrak{F}_{(0,\ \tau_1 \wedge t)}^{M^g}$, we have $\mathfrak{F}_{(0,\ \tau_1 \wedge t)}^{|B|} = \mathfrak{F}_{(0,\ \tau_1 \wedge t)}^{M^f}$.

Now set
$$h(x) = \begin{cases} f(x) & x \geq 0 \\ -f(x) & x \leq 0 . \end{cases}$$

Then $C_h = C_f - \{0\}$, and $\gamma C_h = \{0\}$, so by the result of section 5,

M^h is equivalent to B. Since $\mathcal{F}^{M^f}_{(\tau_1 \wedge t,\ \tau_2 \wedge t)} = \mathcal{F}^{M^h}_{(\tau_1 \wedge t,\ \tau_2 \wedge t)} =$ $\mathcal{F}^{B}_{(\tau_1 \wedge t,\ \tau_2 \wedge t)}$, we obtain: 1) τ_2 is an $\{\mathcal{F}^{M^f}_t\}_{t \geq 0}$-stopping time; and 2) $\mathcal{F}^{|B|}_{(\tau_1 \wedge t,\ \tau_2 \wedge t)} \subset \mathcal{F}^{M^f}_{(\tau_1 \wedge t,\ \tau_2 \wedge t)}$.

Similar arguments yield: for all n τ_n is an $\{\mathcal{F}^{M^f}_t\}_{t \geq 0}$-stopping time, and $\mathcal{F}^{|B|}_{(\tau_n \wedge t,\ \tau_{n+1} \wedge t)} \subset \mathcal{F}^{M^f}_{(\tau_n \wedge t,\ \tau_{n+1} \wedge t)}$. Lemma 3 allows us to conclude: $\mathcal{F}^{|B|}_t \subset \mathcal{F}^{M^f}_t$ for all $t \geq 0$.

3) Finally suppose $C_f = \{z_0\} \cup \gamma^{-1}(0)$, and $\gamma(z_0) = \infty$. Let $\tau_1 = \inf(t:B_t = z_0)$. Then $\mathcal{F}^{M^f}_{(0,\ \tau_1 \wedge t)} = \mathcal{F}^{B}_{(0,\ \tau_1 \wedge t)}$. Applying the results of 2) to the function $g(x) = f(x) - z_0$ and the Brownian Motion $X_s = B_{\tau_1 + s} - B_{\tau_1}$ allow us to conlude that $\mathcal{F}^{|B_i - z_0|}_t \subset \mathcal{F}^{M^f}_t$; and coupled with the result in part a), we get that M^f is equivalent to the Brownian Motion $\{Y_t = \int_0^t \operatorname{sgn}(B_s - z_0)\, dB_s\}_{t \geq 0}$.

<u>Case 2</u>: <u>Card $\gamma^{-1}\{\infty\} > 1$.</u>

Let $z_1 = \sup\{z:z \in (-\infty, 0] \cap \gamma^{-1}\{\infty\}\}$ and $z_2 = \inf\{z:z \in (0, \infty) \cap \gamma^{-1}\{\infty\}\}$, and set $d = z_2 - z_1$. Then z_1 and z_2 are finite, f is periodic with period $2d$, and $\gamma^{-1}\{\infty\} = \{z_1 + kd,\ k = 0, \pm 1, \pm 2, \ldots\}$.

If $z = z_1 + 2kd \pm r$, k an integer, $r \in [0, d]$, say $z \bmod f = r$. Let $\tau_1 = \inf(t:B_t = z_1$ or $B_t = z_2)$ and define for $t \geq 0$
$X_t = B_t 1_{(\tau_1 > t)} + (B \bmod f) \cdot 1_{(\tau_1 \leq t)}$,

a) <u>$\mathcal{F}^{M^f}_t \subset \mathcal{F}^X_t$ for all $t \geq 0$.</u>

Suppose h is continuously differentiable and $\gamma_h^{-1}\{\infty\} = \gamma_f^{-1}\{\infty\}$. Let $H(x) = \int_{z_1}^{x} h(y)\, dy$ for $x \in [z_1, z_1 + 2d]$, and extend H by periodicity. There exist functions G and g defined on $[0, d]$ such that for all $H(z) = G(z \bmod h)$ and $H''(z) = g(z \bmod h)$. Applying Ito's formula, $M_t^h = G(B_t \bmod h) - G(0 \bmod h) - \frac{1}{2}\int_0^t g(B_s \bmod h)\, ds$. Thus $\mathcal{F}_t^{M^h} \subset \mathcal{F}_t^X$. As in Case 1, we may approximate continuous f with continuously differentiable h satisfying $\gamma_f^{-1}\{\infty\} = \gamma_n^{-1}\{\infty\}$, and so $\mathcal{F}_t^{M^f} \subset \mathcal{F}_t^X$ for all $t \geq 0$.

b) $\underline{\mathcal{F}_t^X \subset \mathcal{F}_t^{M^f}}$ __for all t.__ Let $\tau_{n+1} = \inf\{t > \tau_n : B_t \in \{\gamma^{-1}\{\infty\} - B(\tau_n)\}\}$, $n = 1, 2, \ldots$. Then $\mathcal{F}_{(0,\ \tau_1 \wedge t)}^{M^f} = \mathcal{F}_{(0,\ \tau_1 \wedge t)}^{B}$ and for all n,

$$\mathcal{F}_{(\tau_n \wedge t,\ \tau_{n+1}\wedge t)}^{M^f} = \mathcal{F}_{(\tau_n \wedge t,\ \tau_{n+1}\wedge t)}^{|B-B(\tau_n)|} = \mathcal{F}_{(\tau_n \wedge t,\ \tau_{n+1}\wedge t)}^{X} \quad \text{by Case 1 above.}$$

Since the only discontinuity of X occurs at τ_1, we may apply Lemma 3 to conclude that $\mathcal{F}_t^X \subset \mathcal{F}_t^{M^f}$ for all t.

c) $\underline{\mathcal{F}_t^X = \mathcal{F}_t^Y}$ __for all__ $\underline{t \geq 0}$, where $Y_t = \int_0^t g(B_s)\, dB_s$,

$$g(x) = \begin{cases} 1 & \text{on } (z_1, z_2) \\ -1 & \text{on } (z_2, z_2 + d) \end{cases}$$ and g is periodic with period $2d$.

Since $\langle Y \rangle_t = t$, Y is a Brownian Motion. (To show that Y is equivalent to X, apply Lemma 3 with the stopping times τ_n as in b).

Thus M^f is equivalent to the Brownian Motion Y, which is itself equivalent to B reflected in the period interval (z_1, z_2).

7. Proof of theorem: $\gamma^{-1}(0, \infty) \neq \emptyset$

The structure of C_f clutters up the argument in this case, but the basic idea is quite simple. If $\gamma^{-1}(0, \infty)$ is not empty, we may define a square-integrable $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$-stopping time τ which is not predictable -- that is, there exists no sequence of $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$-stopping times $\{\tau_n\}_{n \geq 0}$ such that for all n, $\tau_n < \tau$ a.s. and $\tau_n \uparrow \tau$. Since on Brownian fields, every stopping time is predictable, $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$ can not be generated by any Brownian Motion. The martingale $\{E(\tau | \mathcal{F}_t^{M^f})\}_{t \geq 0}$ has discontinuous paths, with positive probability, so there is no integral representation theorem for the L^2-martingales on $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$.

Case 1 gives the argument in the simplest possible case, while Case 2 indicates how to modify the proof in Case 1 to deal with more complicated crossing sets.

Case 1: $C_f = \{0\}$ and $0 < \gamma(0) < \infty$.

Set $\tau_1 = \inf(t: |B_t| = \gamma(0))$ and $\tau_2 = \inf(t: B_t = \gamma(0))$. If

$$g(x) = \begin{cases} f(x) & x \text{ in } [-\gamma(0), \gamma(0)] \\ f(\gamma(0)) & x \text{ in } [\gamma(0), \infty] \\ -f(\gamma(0)) & x \text{ in } (-\infty, \gamma(0)), \end{cases}$$

then M^g is equivalent to $|B|$, and $\mathcal{F}^{M^f}_{(0, \tau_1)} = \mathcal{F}^{M^g}_{(0, \tau_1)} = \mathcal{F}^{|B|}_{(0, \tau_1)}$. Thus τ_1 is an $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$-stopping time. By lemma 1, for all $t \geq 0$, $|f(B_t)|$ is $\mathcal{F}_t^{M^f}$-measurable, and by lemmas 1 and 2, $|B_t|$ is $\mathcal{F}_t^{M^f}$-measurable also. Hence, by the argument in section 5, for all t, $A = \{\omega: \tau_1(\omega) < t$ and $B(\tau_1) = \gamma(0)\} = \{\tau_2 < t\}$ is in $\mathcal{F}_t^{M^f}$. By the right continuity of $\mathcal{F}_t^{M^f}$, $\{\tau_2 \leq t\} \in \mathcal{F}_t^{M^f}$, and so τ_2 is an $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$-stopping time.

Suppose σ is an $\{\mathcal{F}_t^{M^f}\}$-stopping time and $\sigma < \tau_2$ a.s. Let $S_1 = \{\omega: \sigma(\omega) > \tau_1(\omega)\}$ and $S_2 = \{\omega$: for all t, $B_t(\omega) = -B_t(\omega^*)$ for some ω^* in $S_1\}$. Then $P(S_2) = P(S_1)$. Suppose $\omega \in S_2$, $\omega^* \in S_1$ as in the definition of S_2. Since τ_1 is an $\{\mathcal{F}_t^{|B|}\}_{t \geq 0}$-stopping time, $\tau_1(\omega) = \tau_1(\omega\ )$. Since $\omega^* \in S_1$, $B_{\tau_1}(\omega^*) = -\gamma(0)$ and so $B_{\tau_1}(\omega) = \gamma(0)$ and therefore $\tau_1(\omega) = \tau_2(\omega)$. So: (1) $(\omega) < \tau_1(\omega)$. But σ is an $|\mathcal{F}_t^{M^f}|_{t \geq 0}$-stopping time, and $\mathcal{F}^{M^f}_{(0,\ \tau_1 \wedge t)} = \mathcal{F}^{|B|}_{(0,\ \tau_1 \wedge t)}$; thus (1) implies that $\sigma(\omega) = \sigma(\omega^*)$. But then $\sigma(\omega^*) < \tau_1(\omega) = \tau_1(\omega^*)$, a contradiction since $\omega^* \in S_1$. So we must have $P(S_2) = 0$; that is, $\sigma < \tau_1$ a.s. Since we can find $C > 0$ such that $P(\tau_2 - \tau_1 > C) > 0$, for any $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$-stopping time $\sigma < \tau_2$ a.s., $P(\tau_2 - \sigma > C) > 0$. Hence τ_2 is not $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$-predictable. Neither is $\tau = \tau_2 \wedge (\tau_1 + 1)$, and τ is square integrable.

Case 2: General C_f.

Define z_1 and z_2 as follows:

i) If $(-\infty, 0] \cap \gamma^{-1}(0, \infty) = \emptyset$, $z_1 = -\infty$. Otherwise $z_1 \in (-\infty, 0] \cap \gamma^{-1}(0,\infty)$ and if $z \in (-\infty, 0] \cap \gamma^{-1}(0, \infty)$, then $z_1 - \gamma(z_1) > z - \gamma(z)$.

ii) If $(0, \infty) \cap \gamma^{-1}(0, \infty) = \emptyset$, $z_2 = \infty$. Otherwise $z_2 \in (0, \infty) \cap \gamma^{-1}(0, \infty)$, and if $z \in (0, \infty) \cap \gamma^{-1}(0, \infty)$, $z_2 + \gamma(z_2) < z + \gamma(z)$.

For finite z_i, let $I_i = (z_i - \gamma(z_i), z_i + \gamma(z_i))$ $i = 1, 2$.

Now we define the appropriate stopping times: let $\sigma = \inf\{t: B_t = z_1$ or $B_t = z_2\}$ and $\tau_1 = \inf\{t > \sigma: |B(t) - B(\sigma)| > \gamma(B(\sigma))\}$ and $\tau_2 = \inf\{t > \sigma: B(t) = B(\sigma) + \gamma(B(\sigma))\}$.

Set $\tau = \begin{cases} \tau_2 \wedge (\tau_2 + 1) & \text{if } z_1 \text{ and } z_2 \text{ are finite} \\ 100 \wedge \tau_2 \wedge (\tau_1 + 1) & \text{if either } z_1 \text{ or } z_2 \text{ are infinite.} \end{cases}$

It can easily (if tediously) be checked using all the results and techniques above, that τ_1 and τ_2 are $\{\mathcal{F}_t^{M^f}\}_{t \geq 0}$-stopping times and that τ_2 is not predictable. Moreover $\{E(\tau | \mathcal{F}_t^{M^f})\}_{t \geq 0}$ is an L^2-martingale whose paths are discontinuous with positive probability.

NOTE - The theorem remains true without the condition $Z_f = C_f$, so long as leb $(Z_f) = 0$. Basically, the only new problem is the following: suppose there exist $\epsilon < x_0 < \infty$, such that $f(x_0) = f(-x)$ for $|x| \le x_0$, and for all $0 \le r \le \epsilon$, $f(-x_0 - r) = -f(-x_0 + r)$ while $f(x_0 - r) = f(x_0 + r)$. In this situation $\gamma_f(0) = x_0$, but $|f|$ is locally symmetric around both $-x_0$ and x_0. However, it is still the case that $B(\tau)$ is $\mathcal{F}_\tau^{M^f}$- measurable, where $\tau = \inf\{t: |B|_t = x_0\}$, and this is exactly the condition needed to construct a discontinuous martingale on $\{\mathcal{F}_t^{M^f}\}_t \ge 0$. The proof is similar in spirit to the proof given above, but rather tedious, and so will be omitted.

BIBLIOGRAPHY

Chung and Welsh (1974), "Meyer's Theorem on Predictability," Zeit für Wehr, Vol. 29 No. 3, 253-256.

Kallianpur (1977), Nonlinear Stochastic Filter Theory (to appear)

McKean (1969), Stochastic Integrals